From (1) and (2), $\cos 2\alpha = \dfrac{(c-a)n}{e^2}$.

From (3), $\sin 2\alpha = \dfrac{-2bn}{e^2}$.

These equations determine α without ambiguity. Substituting for α, e, n in the equations

$$h - e^2 p\cos\alpha = -dn,$$
$$k - e^2 p\sin\alpha = -en,$$
$$h^2 + k^2 - e^2 p^2 = fn,$$

and solving for h, k, p, the curve is completely determined.

The solution of these last equations will be much simpler if the given equation is first transformed to the center, for we will then have

$$h = e^2 p\cos\alpha,\ \ k = e^2 p\sin\alpha,\ \ h^2 + k^2 - e^2 p^2 = f'n,$$

f' being obtained by substituting the coördinates of the center in the left hand member of the given equation.

If a'=semi-major axis, b'=semi-minor axis, we have $a' = ep$, $b' = a'\sqrt{(1-e^2)}$.

Let us take the equation $3x^2 + 2xy + 3y^2 - 16y + 20 = 0$.

Transformed to centre $(-1, 3)$, $3x^2 + 2xy + 3y^2 - 4 = 0$, we find

$$e^2 = \tfrac{1}{2},\ \ n = \frac{2-\frac{1}{2}}{6} = \tfrac{1}{4},\ \sin 2\alpha = -1,\ \cos 2\alpha = 0.$$

$$\alpha = 135^\circ,\ \ h = -\frac{p}{\sqrt{8}},\ \ k = \frac{p}{\sqrt{8}},\ \ p = \pm 2,\ \ a' = \sqrt{2},\ \ b' = 1.$$

In general the equations for h, k, p give two values of each quantity showing that the conic has two directrices and two foci.

If $e=1$, then $h - p\cos\alpha = -dn$, $k = -p\sin\alpha = -en$, $h^2 + k^2 - p^2 = fn$. The terms containing p^2 cancel, showing that the parabola has one directrix and one focus at infinity.

INTEGRATION OF ELLIPTIC INTEGRALS.

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[Continued from April Number.]

$$\therefore\ B_0 = \frac{1}{\pi}\int_0^{2\pi} \frac{d\varphi}{(1 + e^2 - 2e\cos\varphi)^{\frac{3}{2}}}$$

$$=\frac{4}{\pi(1-e^2)^2}[2E(e, \tfrac{1}{2}\pi)-(1-e^2)F(e, \tfrac{1}{2}\pi)] \ldots\ldots (64).$$

$$B_1=\frac{1}{\pi}\int_0^{2\pi}\frac{\cos\varphi d\varphi}{(1+e^2-2e\cos\varphi)^{\frac{3}{2}}}$$

$$=\frac{4}{\pi e(1-e^2)^2}[(1+4e^2)E(e, \tfrac{1}{2}\pi)-(1+2e^2)F(e, \tfrac{1}{2}\pi)] \ldots (65).$$

$$C_0=\frac{1}{\pi}\int_0^{2\pi}\frac{d\varphi}{(1+e^2-2e\cos\varphi)^{\frac{5}{2}}}$$

$$=\frac{4}{3\pi(1-e^2)^4}[8(1+e^2)E(e, \tfrac{1}{2}\pi)-(5-2e^2-3e^4)F(e, \tfrac{1}{2}\pi)] \ldots\ldots (66).$$

$$C_1=\frac{1}{\pi}\int_0^{2\pi}\frac{\cos\varphi d\varphi}{(1+e^2-2e\cos\varphi)^{\frac{5}{2}}}$$

$$=\frac{4}{3\pi e(1-e^2)^4}[(1+14e^2-e^4)E(e, \tfrac{1}{2}\pi)-(1+6e^2-7e^4)F(e, \tfrac{1}{2}\pi)] \ldots (67).$$

$$D_0=\frac{1}{\pi}\int_0^{2\pi}\frac{d\varphi}{(1+e^2-2e\cos\varphi)^{\frac{7}{2}}}=\frac{4}{15\pi(1-e^2)^6}[2(23+82e^2$$

$$+23e^4)E(e, \tfrac{1}{2}\pi)-(31+51e^2-67e^4-15e^6)F(e, \tfrac{1}{2}\pi)] \ldots (68).$$

These expressions are also useful in finding the *disturbing* force between two planets.

Before dealing with general expressions we will find expressions containing the Elliptic Integral of the third order.

$$\int_0^{2\pi}\frac{d\theta}{(1+c\sin^2\theta)\sqrt{(1-e^2\sin^2\theta)}}=\Pi(e, c, \tfrac{1}{2}\pi) \ldots\ldots\ldots\ldots\ldots (69).$$

$$\int_0^{\frac{1}{2}\pi}\frac{\sin^2\theta d\theta}{(1+c\sin^2\theta)\sqrt{(1-e^2\sin^2\theta)}}=\frac{1}{c}\int_0^{\frac{1}{2}\pi}\frac{[(1+c\sin^2\theta)-1]d\theta}{(1+c\sin^2\theta)\sqrt{(1-e^2\sin^2\theta)}}$$

$$=\frac{1}{c}\int_0^{\frac{1}{2}\pi}\frac{d\theta}{\sqrt{(1-e^2\sin^2\theta)}}-\frac{1}{c}\int_0^{\frac{1}{2}\pi}\frac{d\theta}{(1+c\sin^2\theta)\sqrt{(1-e^2\sin^2\theta)}}$$

$$=\frac{1}{c}[F(e, \tfrac{1}{2}\pi)-\Pi(e, c, \tfrac{1}{2}\pi)] \ldots\ldots\ldots\ldots (70).$$

$$\int_0^{\frac{1}{2}\pi}\frac{\cos^2\theta d\theta}{(1+c\sin^2\theta)\sqrt{(1-e^2\sin^2\theta)}}=\int_0^{\frac{1}{2}\pi}\frac{(1-\sin^2\theta)d\theta}{(1+c\sin^2\theta)\sqrt{(1-e^2\sin^2\theta)}}$$

$$=\frac{1}{c}[(c+1)\Pi(e,\ c,\ \tfrac{1}{2}\pi)-F(e,\ \tfrac{1}{2}\pi)]\ \ldots\ldots\ldots\ldots(71).$$

$$\int_0^{\frac{1}{2}\pi}\frac{\sin^4\theta d\theta}{(1+c\sin^2\theta)\sqrt{(1-e^2\sin^2\theta)}}=\frac{1}{c^2}\int_0^{\frac{1}{2}\pi}\frac{[(1+c\sin^2\theta)^2-2(1+c\sin^2\theta+1]d\theta}{(1+c\sin^2\theta)\sqrt{(1-e^2\sin^2\theta)}}$$

$$=\frac{1}{c^2}\int_0^{\frac{1}{2}\pi}\frac{d\theta}{(1+c\sin^2\theta)\sqrt{(1-e^2\sin^2\theta)}}+\frac{1}{c^2}\int_0^{\frac{1}{2}\pi}\frac{(c\sin^2\theta-1)d\theta}{\sqrt{(1-e^2\sin^2\theta)}}$$

$$=\frac{1}{c^2e^2}[(c-e^2)F(e,\ \tfrac{1}{2}\pi)-cE(e,\ \tfrac{1}{2}\pi)+e^2\Pi(e,\ c,\ \tfrac{1}{2}\pi)]\ \ldots\ldots\ldots\ (72).$$

$$\int_0^{\frac{1}{2}\pi}\frac{\cos^4\theta d\theta}{(1+c\sin^2\theta)\sqrt{(1-e^2\sin^2\theta)}}=\int_0^{\frac{1}{2}\pi}\frac{(1-2\sin^2\theta+\sin^4\theta)d\theta}{(1+c\sin^2\theta)\sqrt{(1-e^2\sin^2\theta)}}$$

$$=\frac{1}{c^2e^2}[(c+1)^2e^2\Pi(e,\ c,\ \tfrac{1}{2}\pi)-cE(e,\ \tfrac{1}{2}\pi)+(c-e^2-2ce^2)F(e,\ \tfrac{1}{2}\pi)]\ldots(73).$$

$$\int_0^{\frac{1}{2}\pi}\frac{\sin^2\theta\cos^2\theta d\theta}{(1+c\sin^2\theta)\sqrt{(1-e^2\sin^2\theta)}}=\int_0^{\frac{1}{2}\pi}\frac{(\sin^2\theta-\sin^4\theta)d\theta}{(1+c\sin^2\theta)\sqrt{(1-e^2\sin^2\theta)}}$$

$$=\frac{1}{c^2e^2}[cE(e,\ \tfrac{1}{2}\pi)+(ce^2+e^2-c)F(e,\ \tfrac{1}{2}\pi)-(c+1)e^2\Pi(e,\ c,\ \tfrac{1}{2}\pi)]\ldots\ldots\ldots\ldots(74).$$

$$\int_0^{\frac{1}{2}\pi}\frac{\sin^6\theta d\theta}{(1+c\sin^2\theta)\sqrt{(1-e^2\sin^2\theta)}}=\frac{1}{c^3}\int_0^{\frac{1}{2}\pi}\frac{[(1+c\sin^2\theta)^3-(1+3c\sin^2\theta+sc^2\sin^4\theta)]d\theta}{(1+c\sin^2\theta)\sqrt{(1-e^2\sin^2\theta)}}$$

$$=\frac{1}{3c^3e^4}[(3e^4-3ce^2+c^2e^2+2c^2)F(e,\ \tfrac{1}{2}\pi)-3e^4\Pi(e,\ c,\ \tfrac{1}{2}\pi)$$

$$-(2c^2+2c^2e^2-3ce^2)E(e,\ \tfrac{1}{2}\pi)]\ldots\ldots\ldots\ldots\ldots\ldots(75).$$

$$\int_0^{\frac{1}{2}\pi}\frac{\sin^4\theta\cos^2\theta d\theta}{(1+c\sin^2\theta)\sqrt{(1-e^2\sin^2\theta)}}=\int_0^{\frac{1}{2}\pi}\frac{(\sin^4\theta-\sin^6\theta)d\theta}{(1+c\sin^2\theta)\sqrt{(1-e^2\sin^2\theta)}}$$

$$=\frac{1}{3c^3e^4}[3e^4(c+1)\Pi(e,\ c,\ \tfrac{1}{2}\pi)+(2c^2-c^2e^2-3ce^2)E(e,\ \tfrac{1}{2}\pi)$$

$$-(3e^4-3ce^2+2c^2-2c^2e^2+3ce^4)F(e,\ \tfrac{1}{2}\pi)]\ \ldots\ldots(76).$$

$$\int_0^{\frac{1}{2}\pi}\frac{\sin^2\theta\cos^4\theta d\theta}{(1+c\sin^2\theta)\sqrt{(1-e^2\sin^2\theta)}}+\int_0^{\frac{1}{2}\pi}\frac{(\sin^2\theta\cos^2\theta-\sin^4\theta\cos^2\theta)d\theta}{(1+c\sin^2\theta)\sqrt{(1-e^2\sin^2\theta)}}$$

$$=\frac{1}{3c^3e^4}[(3e^4+3c^2e^4-3ce^2+6ce^4-5c^2e^2+2c^2)F(e,\ \tfrac{1}{2}\pi)$$

$$-3e^4(c+1)^2\Pi(e,\ c,\ \tfrac{1}{2}\pi)-(2c^2-4c^2e^2-3ce^2)E(e,\ \tfrac{1}{2}\pi)]\ldots\ldots\ldots(77).$$

$$\int_0^{\frac{1}{2}\pi}\frac{\cos^6\theta d\theta}{(1+c\sin^2\theta)\surd(1-e^2\sin^2\theta)}=\int_0^{\frac{1}{2}\pi}\frac{(\cos^4\theta-\sin^2\theta\cos^4\theta)d\theta}{(1+c\sin^2\theta(\surd(1-e^2\sin^2\theta)}$$

$$=\frac{1}{3\,c^3e^4}[3e^4(c+1)^3\Pi(e,\ c,\ \tfrac{1}{2}\pi)+(2c^2-7c^2e^2-3ce^2)E(e,\ \tfrac{1}{2}\pi)$$

$$-(3e^4+9c^2e^4-3ce^2+9ce^4-8c^2e^2+2c^2)F(e,\ \tfrac{1}{2}\pi)]\ldots\ \ldots\ldots(78).$$

$$\int_0^{\frac{1}{2}\pi}\frac{\sin^8\theta d\theta}{(1+c\sin^2\theta)\surd(1-e^2\sin^2\theta)}$$

$$=\frac{1}{c^4}\int_0^{\frac{1}{2}\pi}\frac{[(1+c\sin^2\theta)^4-(1+4c\sin^2\theta+6c^2\text{ain}^4\theta+4c^3\sin^6\theta)]d\theta}{(1+c\sin^2\theta)\surd(1-e^2\sin^2\theta)}$$

$$=\frac{1}{c^4}\int_0^{\frac{1}{2}\pi}\frac{(1+c\sin^2\theta)^3d\theta}{\surd(1-e^2\sin^2\theta)}-\frac{1}{c^4}\int_0^{\frac{1}{2}\pi}\frac{(1+4c\sin^3\theta+6c^2\sin^4\theta+4c^3\sin^6\theta)d\theta}{(1+c\sin^2\theta)\surd(1-e^2\sin^2\theta)}$$

$$=\frac{1}{15c^4e^6}[(8c^3+3c^3e^2+4c^3e^4-10c^2e^2-5c^2e^4+15ce^4-15e^6)F(e,\ \tfrac{1}{2}\pi)$$

$$+15e^6\Pi(e,\ c,\ \tfrac{1}{2}\pi)-(8c^3+7c^3e^2+8c^3e^4-10c^2e^2-10c^2e^4+15ce^4)E(e,\ \tfrac{1}{2}\pi)]\,..(79).$$

$$\int_0^{\frac{1}{2}\pi}\frac{\sin^6\theta\cos^2\theta d\theta}{(1+c\sin^2\theta)\surd(1-e^2\sin^2\theta)}=\int_0^{\frac{1}{2}\pi}\frac{(\sin^6\theta-\sin^8\theta)d\theta}{(1+c\sin^2\theta)\surd(1-e^2\sin^2\theta)}$$

$$=\frac{1}{15c^4e^6}[(8c^3-3c^3e^2-2c^3e^4-10c^2e^2+5c^2e^4+15ce^4)E(e,\ \tfrac{1}{2}\pi)$$

$$-15e^6(c+1)\Pi(e,\ c,\ \tfrac{1}{2}\pi)-(8c^3-7c^3e^2-c^3e^4-10c^2e^2$$

$$+10c^2e^4+15ce^4-15ce^6-15e^6)F(e,\ \tfrac{1}{2}\pi)]\ldots\ldots\ldots\ldots\ldots(80).$$

$$\int_0^{\frac{1}{2}\pi}\frac{\sin^4\theta\cos^4\theta d\theta}{(1+c\sin^2\theta)\surd(1-e^2\sin^2\theta)}=\int_0^{\frac{1}{2}\pi}\frac{(\sin^4\theta\cos^2\theta-\sin^6\theta\cos^2\theta)d\theta}{(1+c\sin^2\theta)\surd(1-e^2\sin^2\theta)}$$

$$=\frac{1}{15c^4e^6}[15e^6(e+1)^2\Pi(e,\ c,\ \tfrac{1}{2}\pi)-(8c^3-13c^3e^2+3c^3e^4-10c^2e^2+20c^2e^4$$

$$+15ce^4)E(e,\ \tfrac{1}{2}\pi)+(8c^3-17c^3e^2+9c^3e^4-10c^2e^2+25c^2e^4$$

$$+15ce^4-30ce^6-15c^2e^6-15e^6)F(e,\ \tfrac{1}{2}\pi)]\ldots\ldots\ldots(81).$$

$$\int_0^{\frac{1}{2}\pi}\frac{\sin^2\theta\cos^6\theta d\theta}{(1+c\sin^2\theta)\surd(1-e^2\sin^2\theta)}=\int_0^{\frac{1}{2}\pi}\frac{(\sin^2\theta\cos^4\theta-\sin^4\theta\cos^4\theta)d\theta}{(1+c\sin^2\theta)\surd(1-e^2\sin^2\theta)}$$

$$=\frac{1}{15c^4e^6}[(8c^3-23c^3e^2+23c^3e^4-10c^2e^2+35c^2e^4+15ce^4)E(e,\ \tfrac{1}{2}\pi)$$

$$-15e^6(c+1)^3\Pi(e,\ c,\ \tfrac{1}{2}\pi)-(8c^3-27c^3e^2+34c^3e^4-10c^2e^2$$

$$+40c^2e^4+15ce^4-15c^3e^6-45c^2e^6-15e^6)F(e,\ \tfrac{1}{2}\pi)]\ldots\ldots\ldots\ldots(82).$$

$$\int_0^{\frac{1}{2}\pi}\frac{\cos^8\theta d\theta}{(1+c\sin^2\theta)\sqrt{(1-e^2\sin^2\theta)}}=\int_0^{\frac{1}{2}\pi}\frac{(\cos^6\theta-\cos^6\theta\sin^2\theta)d\theta}{(1+c\sin^2\theta)\sqrt{(1-e^2\sin^2\theta)}}$$

$$=\frac{1}{15c^4e^6}[15e^6(c+1)^4\Pi(e,\ c,\ \tfrac{1}{2}\pi)-(8c^3-33c^3e^2+58c^3e^4-10c^2e^2+50c^2e^4$$

$$+15ce^4)E(e,\ \tfrac{1}{2}\pi)+(8c^3-37c^3e^2+74c^3e^4-10c^2e^2+55c^2e^4$$

$$+15ce^4-60c^3e^6-90c^2e^6-60ce^6-15e^6)F(e,\ \tfrac{1}{2}\pi)]\ldots\ldots\ldots\ldots(83).$$

$$\int_0^{\frac{1}{2}\pi}\frac{\sin^{10}\theta d\theta}{(1+c\sin^2\theta)\sqrt{(1-e^2\sin^2\theta)}}=\frac{1}{c^5}\int_0^{\frac{1}{2}\pi}\frac{(1+c\sin^2\theta)^4d\theta}{\sqrt{(1-e^2\sin^2\theta)}}$$

$$-\frac{1}{c^5}\int_0^{\frac{1}{2}\pi}\frac{(1+5c\sin^2\theta+10c^2\sin^4\theta+10c^3\sin^6\theta+5c^4\sin^8\theta)d\theta}{(1+c\sin^2\theta)\sqrt{(1-e^2\sin^2\theta)}}$$

$$=\frac{1}{105c^5e^8}[(48c^4+16c^4e^2+17c^4e^4+24c^4e^6-56c^3e^2-21c^3e^4-28c^3e^6$$

$$+70c^2e^4+35c^2e^6-105ce^6+105e^8)F(e,\ \tfrac{1}{2}\pi)-(48c^4+40c^4e^2$$

$$+40c^4e^4+48c^4e^6-56c^3e^2-49c^3e^4-56c^3e^6+70c^2e^4+70c^2e^6$$

$$-105ce^6)E(e,\ \tfrac{1}{2}\pi)-105e^8\Pi(e,\ c,\ \tfrac{1}{2}\pi)]\ldots\ldots\ldots\ldots(84).$$

As the foregoing are ample for illustration, we will proceed to other considerations.

[To be Continued.]